POWER LAW INFLATION AND THE COSMIC NO HAIR THEOREM IN BRANE WORLD

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Abstract

The Cosmic no hair theorem is studied in anisotropic Bianchi brane models which admit power law inflation with a scalar field. We note that all Bianchi models except Bianchi type IX transit to an inflationary regime and the anisotropy washes out at a later epoch. It is found that in the brane world, the anisotropic universe approaches the isotropic phase via inflation much faster than that in the general theory of relativity. The modification in the Einstein field equations on the brane is helpful for a quick transition to an isotropic era from the anisotropic brane. We note a case where the curvature term in the field equation initially drives power law inflation on the isotropic brane which is however not permitted without the brane framework.

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During the last couple of years there has been a growing interest to study cosmological models in higher dimensions motivated by the developments in superstring and M-theory [1,2]. These theories may be considered as promising candidates for a quantum theory of gravity. Although a complete theory of quantum gravity is yet to emerge, it is interesting to look for cosmological issues in the string theories. In the above, one requires dimensions more than four for a consistent formulation. It is therefore, interesting to probe cosmological issues in this framework. The work on higher dimensional cosmology began with the work of Kaluza-Klein (KK) [3]. In the past, the usual four dimensional space-time was recovered starting with a higher dimensional one either by a dimensional reduction mechanism or by considering compact extra space dimensions which are not visible today. Recently, the above ideas have been changed remarkably. According to recent views, in higher dimensional scenario, our observed universe may be described by a brane embedded in higher dimensional space-time with the usual matter fields and force confined on the brane. The gravitational field may propagate through the bulk dimensions perpendicular to the brane. Randall and Sundrum [2] shown that even if the extra dimensions are not compact, in the brane world model one recovers four dimensional Newtonian gravity starting with a five dimensional anti-de Sitter spacetime (ADS_5) in the low energy limit.

Recently there has been a spurt in activities in building cosmological models of the very early universe in the brane world [4]. It is now generally believed that inflation is one of the essential ingredients in modern cosmology as it can solve some of the problems of the big bang model elegantly. Two varieties of inflationary solutions have been proposed in the literature : (i) exponential or quasi-exponential expansion and (ii) power law inflation. The first kind of inflation requires a potential which behaves as a cosmological constant and in the second case a power law inflation is obtained by a Salam-Sezgin type exponential potential. The possibility of a chaotic inflation scenario in the brane world was studied by

Maartens *et al.* [5] and it was found that the modified braneworld Friedmann equation leads to a stronger condition for inflation. The brane effects ease the condition for slow-roll inflation for a massive scalar field. It was also shown [6] that in the framework of a self-interacting quartic type potential, a chaotic model may be realised and the initial condition $\phi > 3M_P$ which is required in the general theory of relativity (GTR) is found to be no longer an essential condition in the braneworld. This is a good feature on branes as chaotic inflation in GTR has been criticised for regarding super Planckian field values and can lead to a non-linear quantum correction in the potential which was ignored. The dynamics of inflaton on the brane due to the high energy brane corrections introduced in the field equations are addressed in the literature [7].

The behavior of an anisotropic Bianchi type-I brane world in the presence of a scalar field with a large anisotropy was explored by Maartens, Sahni and Saini [8]. It is shown that a large anisotropy enhances more damping into the scalar field equation of motion, resulting in greater inflation. In the last couple of years brane models with anisotropic universes exploring different aspects of the early universe have been reported in the literature [8-12]. In the brane world the validity of the cosmic no hair theorem for global anisotropy with [13] or without [14] four dimensional cosmological constant has been explored. However, in the latter case the scalar field potential behaves as an effective four dimensional cosmological constant. Cosmological models with power law inflation in the brane world are of recent interest [15]. It is also important to explore power law inflation in the anisotropic brane world and look for its isotropization. In four dimensions Kitada and Maeda [16] extended Wald's idea [17] of the cosmic no hair theorem for scalar fields with an exponential potential which admits power law inflation in Bianchi universes. In this paper the cosmic no hair theorem for power law inflation is studied with exponential potential in anisotropic Bianchi brane models.

The Einstein field equations in the five dimensional (bulk) space-time is given by

$$G_{AB}^{(5)} = \tilde{\kappa}^2 \left[-g_{AB}^{(5)} \Lambda_{(5)} + T_{AB}^{(5)} \right] \tag{1}$$

with $T_{AB}^{(5)} = \delta(y)[-\lambda g_{AB} + T_{AB}]$. Here $\tilde{\kappa}$ represents the five dimensional gravitational coupling constant, $g_{AB}^{(5)}$, $G_{AB}^{(5)}$ and $\Lambda_{(5)}$ are the metric, Einstein tensor and the cosmological constant of the bulk space-time respectively, T_{AB} is the matter energy momentum tensor. We have $\tilde{\kappa}^2 = \frac{8\pi}{M_P^3}$, where $M_P = 1.2 \times 10^{19}$ GeV. A natural choice of coordinates is $x^A = (x^\mu, y)$ where $x^\mu = (t, x^i)$ are space-time coordinates on the brane. The upper case Latin letters $(A, B, ... = 0, ..., 4)$ represent coordinate indices in the bulk spacetime, the Greek letters $(\mu, \nu, ... = 0, ..., 3)$ the coordinate indices in the four dimensional spacetime and the small case latin letters $(i, j = 1, 2, 3)$ the three space. The space-like hypersurface $x^4 = y = 0$ gives the brane world and g_{AB} is its induced metric, λ is the tension of the brane which is assumed to be positive in order to recover the conventional general theory of gravity (GTR) on the brane. The bulk cosmological constant $\Lambda_{(5)}$ is negative and represents the five dimensional cosmological constant.

The field equations induced on the brane are derived by Shiromizu *et al* [18] using a geometric approach which leads to new terms carrying bulk effects on the brane. The modified dynamical equations on the brane are

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \tilde{\kappa}^4 S_{\mu\nu} - E_{\mu\nu}. \tag{2}$$

The effective cosmological constant Λ and the four dimensional constant κ on the brane are given by

$$\Lambda = \frac{|\Lambda_5|}{2} \left[\left(\frac{\lambda}{\lambda_c} \right)^2 - 1 \right],$$

$$\kappa^2 = \frac{1}{6} \lambda \, \tilde{\kappa}^4 \tag{3}$$

respectively, where λ_c is the critical brane tension which is given by

$$\lambda_c = 6\frac{|\Lambda_5|}{\tilde{\kappa}^2}. \tag{4}$$

However one can make the effective four dimensional cosmological constant zero by a choice of the brane tension. The extra dimensional corrections to the Einstein equations on the brane are of two types and are given by :

- $S_{\mu\nu}$: quadratic in the matter variables which is

$$S_{\mu\nu} = \frac{1}{12}TT_{\mu\nu} - \frac{1}{4}T_{\mu\alpha}T^{\alpha}_{\nu} + \frac{1}{24}g_{\mu\nu}\left[3T_{\alpha\beta}T_{\alpha\beta} - (T^{\alpha}_{\alpha})^2\right]. \tag{5}$$

where $T = T^{\alpha}_{\alpha}$, $S_{\mu\nu}$ is significant at high energies i.e., $\rho > \lambda$,

- $E_{\mu\nu}$: occurs due to the non-local effects from the free gravitational field in the bulk, which enters the equation via the projection $\mathbf{E}^{(5)}_{AB} = C^{(5)}_{ACBD}n^C n^D$ where n^A is normal to the surface ($n^A n_A = 1$). The term is symmetric and traceless and without components orthogonal to the brane, so $\mathbf{E}_{AB}n^B = 0$ and $\mathbf{E}_{AB} \rightarrow E_{\mu\nu}g^{\mu}_{A}g^{\nu}_{B}$ as $y \rightarrow 0$.

To anlyse the cosmological evolution, we consider two components of the dynamical equation (2). First we consider the "initial-value" constraint equation

$$G_{\mu\nu}n^{\mu}n^{\nu} = \kappa^2 T_{\mu\nu}n^{\mu}n^{\nu} + \tilde{\kappa}^4 S_{\mu\nu}n^{\mu}n^{\nu} - E_{\mu\nu}n^{\mu}n^{\nu} \tag{6}$$

and the Raychaudhuri equation

$$R_{\mu\nu}n^{\mu}n^{\nu} = \kappa^2\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)n^{\mu}n^{\nu} + \tilde{\kappa}^4\left(S_{\mu\nu} - \frac{1}{2}g_{\mu\nu}S\right)n^{\mu}n^{\nu} - E_{\mu\nu}n^{\mu}n^{\nu} \tag{7}$$

where n^{μ} is the unit normal to the homogeneous hypersurface. It may be pointed out here that both $G_{\mu\nu}n^{\mu}n^{\nu}$ and $R_{\mu\nu}n^{\mu}n^{\nu}$ are expressed in terms of the three geometry of the homogeneous hypersurfaces and the extrinsic curvature $K_{\mu\nu} = \bigtriangledown_{\nu}n_{\mu}$ respectively. The extrinsic curvature can be decomposed into its trace K and trace-free part $\sigma_{\mu\nu}$ which represents the shear of the timelike geodesic congruence orthogonal to the homogeneous

hypersurface

$$K_{\mu\nu} = \frac{1}{3} K h_{\mu\nu} + \sigma_{\mu\nu}$$

where $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ projects orthogonal to n_μ.

We consider a scalar field theory to describe the energy momentum tensor which is given by

$$T_{\mu\nu} = \phi,_\mu \phi,_\nu - g_{\mu\nu}\left[\frac{1}{2} g^{\alpha\beta}\phi,_\alpha \phi,_\beta + V(\phi)\right] \tag{8}$$

with $V(\phi) = V_o e^{-\eta\tilde{\kappa}^2\phi}$, where η and V_o are constants. For a homogeneous scalar field the dynamical equation (6) and (7) can now be written as

$$K^2 = 3\kappa^2\left[\frac{1}{2}\phi'^2 + V(\phi)\right] + \frac{\tilde{\kappa}^4}{4}\left[\frac{1}{2}\phi'^2 + V(\phi)\right]^2 + \frac{3}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} - \frac{3}{2}\,{}^{(3)}R - 3E_{\mu\nu}n^\mu n^\nu, \tag{9}$$

$$K' = \kappa^2\left[-\phi'^2 + V(\phi)\right] - \frac{\tilde{\kappa}^4}{12}\left[\frac{1}{2}\phi'^2 + V(\phi)\right]\left[\frac{5}{2}\phi'^2 - V(\phi)\right] - \frac{1}{3}K^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + E_{\mu\nu}n^\mu n^\nu \tag{10}$$

where the prime represents differentiation w.r.t cosmic time t. The wave equation for the scalar field is given by

$$\phi'' + 3H\phi' = -\frac{dV(\phi)}{d\phi} \tag{11}$$

with ${}^{(3)}R$ as the scalar curvature of the homogeneous hypersurface. To study the cosmic no hair theorem, we follow Kitada and Maeda [16]. Accordingly we use a new time coordinate τ , which is defined by

$$d\tau = exp(-\eta\tilde{\kappa}^2\phi/2)\ dt \tag{12}$$

instead of the cosmic time t. This conformal change in the time scale is considered here as in an isotropic and homogeneous space time i.e., power law inflation becomes a time independent fixed point, the attractor. The equations (9) and (10) can be rewritten with this time as

$$\tilde{K}^2 = 3\kappa^2\left[\frac{1}{2}\dot{\phi}^2 + V_o\right]e^{\eta\tilde{\kappa}^2\phi} + \frac{\tilde{\kappa}^4}{4}\left[\frac{1}{2}\dot{\phi}^2 + V_o\right]^2 + \frac{3}{2}\tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu} - \frac{3}{2}\,{}^{(3)}\tilde{R} - 3\tilde{E}_{\mu\nu}n^\mu n^\nu, \tag{13}$$

$$\dot{\tilde{K}} = \lambda\tilde{\kappa}^2\dot{\phi}\tilde{K} + \kappa^2\left[-\dot{\phi}^2 + V_o\right]e^{\eta\tilde{\kappa}^2\phi} - \frac{\tilde{\kappa}^4}{12}\left[\frac{1}{2}\dot{\phi}^2 + V_o\right]\left[\frac{5}{2}\dot{\phi}^2 - V_o\right] - \frac{1}{3}\tilde{K}^2 - \tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu} + \tilde{E}_{\mu\nu}n^\mu n^\nu \tag{14}$$

where the over dot indicates a derivative with respect to the new time scale and $\tilde{K} = Ke^{\eta\tilde{\kappa}^2\phi}$, $\tilde{\sigma}_{\mu\nu} = \sigma_{\mu\nu}e^{\eta\tilde{\kappa}^2\phi}$, ${}^{(3)}\tilde{R} =^{(3)} Re^{2\eta\tilde{\kappa}^2\phi}$, $\tilde{E_{\mu\nu}} = E_{\mu\nu}e^{2\eta\tilde{\kappa}^2\phi}$. The scalar field equation becomes

$$\ddot{\phi} = \frac{1}{2}\lambda\tilde{\kappa}^2\dot{\phi}^2 - \tilde{K}\dot{\phi} + \lambda\tilde{\kappa}^2 V_o. \tag{15}$$

In Bianchi universes except for Bianchi type IX, the three curvature is negative [17] i.e., we have

$${}^{(3)}\tilde{R} \leq 0.$$

Now using the above inequality, it is evident from equation (13) that the constraint equation leads to the inequality $\tilde{K}^2 > 0$ i.e., $\tilde{K} > 0$ (i.e., it will expand for ever) if the space-time is initially expanding satisfying the constraint

$$\tilde{E}_{\mu\nu}n^\mu n^\nu \leq 0. \tag{16}$$

In this paper we consider zero contribution for the dark energy. So, we have

$$\tilde{K}^2 > 3\kappa^2 e^{\eta\tilde{\kappa}^2\phi}\left[\frac{1}{2}\dot{\phi}^2 + V_o\right] + \frac{\tilde{\kappa}^4}{4}\left[\frac{1}{2}\dot{\phi}^2 + V_o\right]^2 \tag{17}$$

for all time t. Moss and Sahni [19] first extended Wald's idea for an anisotropic universe with cosmological constant. To study the cosmic no hair theorem with exponential potential, it is required to define a term introduced by Wald [17] and subsequently for power law inflation by Kitada and Maeda [16]. Let us now define a new term in the brane world scenario which is

$$K_\phi = \tilde{K} - \frac{\tilde{\kappa}^2}{2}\left(\frac{1}{2}\dot{\phi}^2 + V_o\right) \tag{18}$$

Using the inequality (17) it is found that K_ϕ is always positive definite. The time differ-

entiation of equation (18) can be written as

$$\dot{K}_\phi = -\frac{1}{3}K_\phi\left[\tilde{K} - \frac{5}{4}\kappa^2\dot{\phi}^2 + \frac{\tilde{\kappa}^2}{2}V_o - 3\eta\tilde{\kappa}^2\dot{\phi}\right] - \tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu} - \kappa^2 e^{\eta\tilde{\kappa}^2\phi}\left[\dot{\phi}^2 - V_o\right]. \qquad (19)$$

As $\tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu} > 0$ and considering only brane tension to dominate we get

$$\dot{K}_\phi \leq -\frac{1}{3}\left[\tilde{K} - \frac{5}{4}\tilde{\kappa}^2\dot{\phi}^2 + \frac{1}{2}\kappa^2 V_o - 3\eta\tilde{\kappa}^2\dot{\phi}\right]K_\phi \qquad (20)$$

when $V_o >> \dot{\phi}^2$. Eq. (20) yields

$$\frac{\dot{K}_\phi}{K_\phi} \leq -\frac{1}{\tau_{iso}} \leq 0 \qquad (21)$$

where $\tau_{iso} = \frac{6}{\tilde{\kappa}^2 V_o}$, which is a constant, depends on the brane tension. On integrating the inequality (21) we get

$$0 \leq K_\phi \leq K_{\phi_o}\exp\left[-\frac{(\tau - \tau_o)}{\tau_{iso}}\right] \leq 0 \qquad (22)$$

where $K_{\phi_o} = K_\phi(\tau = \tau_o)$ and K_ϕ decays exponentially with respect to time τ.

The expansion rate of the universe is dominated by the inflaton energy density. The shear ($\tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu}$),and the three curvature (${}^{(3)}\tilde{R}$) are decreasing rapidly leading to vanishing magnitude. In particular, the decay in anisotropy leads to an isotropic and homogeneous space-time. It is observed that the new time coordinate τ depends on the scalar field, or in other words the isotropization time scale depends on ϕ. To understand the isotropization in term of the cosmic time scale, the equations of motion are to be solved. As an isotropic power law inflation can be realized rapidly in the τ-time coordinate, one may estimate it using the isotropic attractor solution. For this we consider a power law solution given by

$$a = a_o t^\alpha, \qquad (23)$$

$$\phi = \phi_o + \frac{2}{\eta\tilde{\kappa}^2}\ln t, \qquad (24)$$

where $a_o,\ \phi_o$ are constants and α is to be determined. Sahni *et al* [15] shown that for a suitable choice of parameter values of the scalar field potential, which is exponential, one

can realize power law inflation on the brane. The extra dimensions in the theory lead to inflation on the brane which however is not capable of sustaining inflation in GTR. We note that an isotropic power law inflation ($a(t) = a_o\, t^2$) is possible on the brane, which is due to the curvature term in the field equations. The initial size of the universe is determined $\left(a_o^2 = \frac{12}{\tilde{\kappa}^2 V_o} \right)$, when the potential energy of the inflaton field dominates in a closed model of the universe. It is evident from eq. (22) that

$$exp \left(- \frac{\tau}{\tau_{iso}} \right) \infty \left(\frac{t}{t_o} \right)^{-\beta} \tag{25}$$

where β is a positive constant which can be determined using (12) and (22). This implies that K, $\sigma_{\mu\nu}\sigma^{\mu\nu}$, ${}^{(3)}R$ varies as $\left(\frac{t}{t_o} \right)^{-(2+\beta)}$ and $K^2 \propto \left(\frac{t}{t_o} \right)^{-2}$. In the brane, the isotropization scale $\tau_{iso} = \sqrt{\frac{6}{\kappa^2} \left(\frac{\lambda}{V_o} \right)}$ is very small as $\frac{V_o}{\lambda} \rightarrow \infty$ at a very high energy scale. In the low energy scale, $\frac{V_o}{\lambda} \rightarrow 0$, thus compared to GTR isotropization takes place rapidly on the brane.

To conclude, it is found that the Bianchi universes (except type IX) permit power law inflation with a scalar field with exponential potential in the brane world. It is also noted that in the brane world scenario, the universe isotropizes faster than that in GTR.

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